- In engineering, linear-time invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI. In
- practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

Section 3.1

Convolution

- The (CT) convolution of the functions x and h, denoted $x*h$, is defined as the function

$$x*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

- The convolution result $x*h$ evaluated at the point t is simply a weighted average of the function x, where the weighting is given by h time reversed and shifted by t.
- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.
- As we shall see, convolution is used extensively in systems theory.
- In particular, convolution has a special significance in the context of LTI systems.

- To compute the convolution

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau,$$

we proceed as follows:

1. Plot $x(\tau)$ and $h(t-\tau)$ as a function of τ.
2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\tau)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $x * h(t)$.
4. Increase t gradually until the expression for $x * h(t)$ changes form. Record the interval over which the expression for $x * h(t)$ was valid.
5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\tau)$ being shifted very far to the right on the time axis.
6. The results for the various intervals can be combined in order to obtain an expression for $x * h(t)$ for all t.

- The convolution operation is *commutative*. That is, for any two functions x and h,

$$x * h = h * x.$$

- The convolution operation is *associative*. That is, for any signals x, h_1, and h_2,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is *distributive* with respect to addition. That is, for any signals x, h_1, and h_2,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

- For any function x,
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau = x*\delta(t).$$
- Thus, any function x can be written in terms of an expression involving δ.
- Moreover, δ is the *convolutional identity*. That is, for any function x,
$$x*\delta = x.$$

- The convolution of two periodic functions is usually not well defined.
- This motivates an alternative notion of convolution for periodic signals known as periodic convolution.
- The periodic convolution of the T-periodic functions x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(t) = \int_T x(\tau)h(t-\tau)\,d\tau,$$

where $\int_T$ denotes integration over an interval of length T.
- The periodic convolution and (linear) convolution of the T-periodic functions x and h are related as follows:

$$x \circledast h(t) = x_0 * h(t) \quad \text{where} \quad x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

(i.e., $x_0(t)$ equals $x(t)$ over a single period of x and is zero elsewhere.)

Section 3.2

Convolution and LTI Systems

- The response h of a system $\mathcal{H}$ to the input δ is called the impulse response of the system (i.e., $h = \mathcal{H}\{\delta\}$).
- For any LTI system with input x, output y, and impulse response h, the following relationship holds:

$$y = x * h.$$

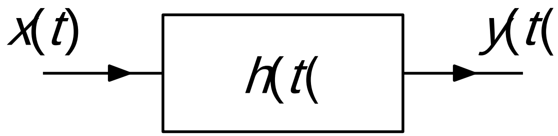
- In other words, a LTI system simply *computes a convolution*.
- Furthermore, a LTI system is *completely characterized* by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.
- Since the impulse response of a LTI system is an extremely useful quantity, we often want to determine this quantity in a practical setting.
- Unfortunately, in practice, the impulse response of a system cannot be determined directly from the definition of the impulse response.

- The response s of a system H to the input u is called the step response of the system (i.e., $s = H\{u\}$).
- The impulse response h and step response s of a system are related as

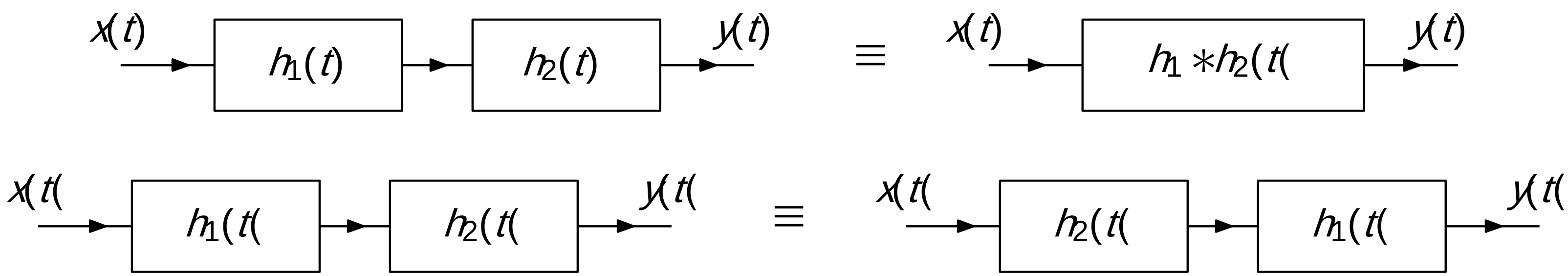
$$h(t) = \frac{ds(t)}{dt}.$$

- Therefore, the impulse response of a system can be determined from its step response by differentiation.
- The step response provides a practical means for determining the impulse response of a system.

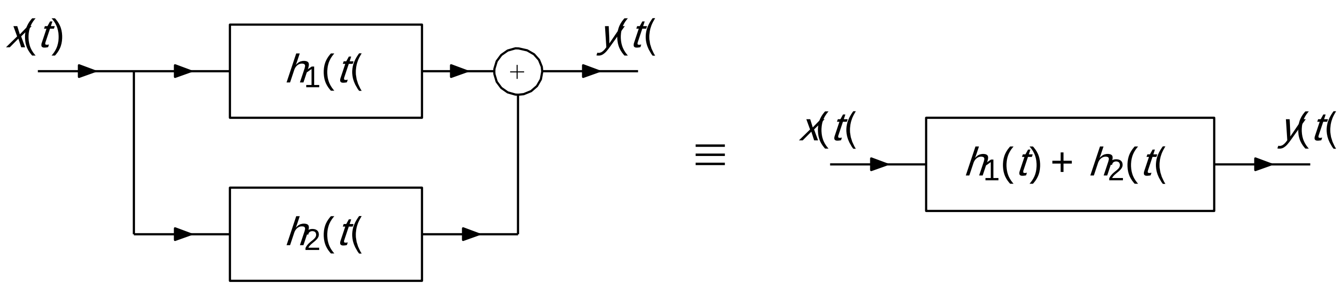
- Often, it is convenient to represent a (CT) LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.



- The *series* interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h = h_1 * h_2$. That is, we have the equivalences shown below.



- The *parallel* interconnection of the LTI systems with impulse responses h_1 and h_2 is a LTI system with the impulse response $h = h_1 + h_2$. That is, we have the equivalence shown below.



Section 3.3

Properties of LTI Systems

- A LTI system with impulse response h is memoryless if and only if

$$h(t) = 0 \quad \text{for all } t \neq 0.$$

- That is, a LTI system is memoryless if and only if its impulse response h is of the form

$$h(t) = K\delta(t)$$

  where K is a complex constant.

- Consequently, every memoryless LTI system with input x and output y is characterized by an equation of the form

$$y = x * (K\delta) = Kx$$

  (i.e., the system is an ideal amplifier).

- For a LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

- A LTI system with impulse response h is causal if and only if

$$h(t) = 0 \quad \text{for all } t < 0$$

)i.e., h is a causal signal.(

- It is due to the above relationship that we call a signal x, satisfying

$$x(t) = 0 \quad \text{for all } t < 0$$

a causal signal.